Notes on Shapes of Polyhedra

Rich Schwartz

June 25, 2015

1 Introduction

Bill Thurston wrote a beautiful paper called *Shapes of Polyhedra*. I once lectured on this paper during a graduate class I taught at the University of Chicago, and recently (Fall 2013) I tried again during my graduate class at ICERM/Brown. I found the paper hard-going both times. In the intervening years, Thurston published an updated and improved version, but I found the new version hard going as well.

I wrote these notes for the ICERM class, and some people (both in and out of class) found them very useful. After some encouragement, I decided to put them on the arXiv, so that they have a public and stable home. I am pretty sure that the proofs are correct, but perhaps I am still missing something. Take them or leave them.

One of the most difficult parts of the paper is the discussion of complex hyperbolic cone manifolds. For one thing, the definition is hard to grasp. For another thing, it is hard to see that the moduli spaces in question really are complex hyperbolic cone manifolds according to the definition. In these notes, I will explain things without relying on cone manifolds at all. Rather, I will introduce related objects which are easier to understand.

At the end of this note, I'll list some other references, one from Curt McMullen and several from John Parker, which treat topics closely related to Thurston's paper.

1.1 Main Results

Part of Thurston's paper deals with triangulations of the sphere and the other part deals with moduli spaces of flat cone spheres. I'll talk about the

flat cone spheres first. A *flat cone sphere* is a metric on the sphere which is locally isometric to the Euclidean plane except at finitely many points, where it has positive conical singularities. This means that a neighborhood of the point is isometric to a Euclidean cylinder with cone angle $2\pi - \theta$ for some $\theta \in (0, 2\pi)$. The number θ is called the cone deficit. One might say that the ordinary points have cone deficit zero.

Let $\theta_1, ..., \theta_m$ be a finite list of positive numbers such that $\sum \theta_i = 4\pi$. Let $\mathcal{M}$ denote the moduli space of similarity classes of flat cone spheres with labeled cone deficits $\theta_1, ..., \theta_m$. Sometimes we shall take the "labeled moduli space", in which all the cone points are labeled. In this moduli space, two flat cone structures are close if there is a near-isometry which maps cone points to cone points and respects the labels. At other times, we shall take the "unlabeled moduli space". In this space, two flat cone structures are close if there is a near isometry between them which maps cone points to cone points and respects the deficit values. If all the cone deficits are distinct, the two spaces are the same. In general, the unlabeled space is a quotient of the labeled space by a finite group of isometries.

Let $\boldsymbol{CH}^n$ denote complex hyperbolic space. The first main result in Thurston's paper is

Theorem 1.1 *Let $\mathcal{M}$ be the labeled moduli space. $\mathcal{M}$ has a natural metric with respect to which it is locally isometric to $\boldsymbol{CH}^{m-3}$.*

Remark: Theorem 1.1 is not quite true in the unlabeled case. For instance, suppose $m = 4$ and $\theta_i = \pi$ for all i. In this case, $\mathcal{M}$ is isometric to the familiar modular orbifold.

The metric on $\mathcal{M}$ is incomplete whenever there are two deficits θ_i and θ_j such that $\theta_i + \theta_j < 2\pi$. Really, the example mentioned in the remark is the only nontrivial example where the metric is complete.

In the incomplete case, which essentially always happens, Thurston goes on to prove (in some sense) that the completion is a complex hyperbolic cone manifold. The most interesting case occurs when the list of deficits satisfies two additional conditions.

1. If $\theta_i + \theta_j < 2\pi$ then 2π is an integer multiple of $2\pi - \theta_i - \theta_j$.
2. If $\theta_i = \theta_j < \pi$ then 2π is an integer multiple of $\pi - \theta_i$.

This case occurs for the flat cone spheres which arise in connection with the triangulations. In this case, Thurston proves a stronger result, one highlight of the paper.

Theorem 1.2 *Let $\mathcal{M}$ be the unlabeled moduli space. If the deficit list satisfies the additional conditions, then there is a lattice Γ acting on $\boldsymbol{CH}^{m-3}$ so that the metric completion of $\mathcal{M}$ is isometric to $\boldsymbol{CH}^{m-3}/\Gamma$.*

Remark: Theorem 1.2 is not quite true for the labeled space. However, Theorem 1.2 is true for the labeled space if all the cone deficits are distinct, or if it never happens that there are two equal cone deficits less than π.

Let **Eis** denote the Eisenstein lattice, $\boldsymbol{Z}[\omega]$, where $\omega = \exp(2\pi i/3)$ is the usual cube root of unity. The points in **Eis** are the vertices of the usual triangulation of the plane by equilateral triangles.

Say that a triangulation of the sphere is *combinatorially positive* if there are never more than 6 triangles around a vertex. Each combinatorial positive triangulation gives rise to a flat cone sphere – one just glues together equilateral triangles in the same pattern. If k triangles go around a vertex, the corresponding deficit is $(6-k)\pi/3$.

Let's think of these triangulations as giving points in the unlabeled moduli space $\mathcal{M}$. We call $\mathcal{M}$ *special* if it contains at least one point corresponding to a triangulation. Call a point in $\mathcal{M}$ a *triangulation point* if it comes from a triangulation. The following result is not explicitly stated in Thurston's paper, but it is implied by other results.

Theorem 1.3 *If $\mathcal{M}$ is special, the set of triangulation points is dense in $\mathcal{M}$. There is a single lattice Γ, acting on $\boldsymbol{C}^{1,9}$, defined over* **Eis**, *such that every special moduli space is isometric to some stratum of $\boldsymbol{CH}^9/\Gamma$.*

The set $\mathbf{Eis}^{1,9}$ denotes the set of vectors in $\boldsymbol{C}^{1,9}$ having coordinates in **Eis**. Here is Thurston's main result about combinatorially positive triangulations, the other highlight of the paper:

Theorem 1.4 *There is a natural bijection between the set of combinatorially positive triangulations and the set of vectors in $\mathbf{Eis}^{1,9}/\Gamma$ having positive square norm with respect to a Γ-invariant Hermitian form H of type $(1,9)$. Here Γ is a lattice defined over* **Eis** *and H is also defined over* **Eis**. *The square norm $H(V,V)$ of a positive vector $V \in \mathbf{Eis}^{1,9}$ is 3 times the number of triangles in the triangulation corresponding to V.*

1.2 Organization of the Notes

I'll explain the proof of Theorem 1.1 in §2. The proof essentially follows Thurston's outline, except that I do some of the details differently.

Thurston proves Theorem 1.2 in three steps.

1. Ignoring Conditions 2 and 3 above, the completion of $\mathcal{M}$ is always a finite volume complex hyperbolic cone manifold. Conditions 2 and 3 together imply that the completion of $\mathcal{M}$ has codimension 2 orbifold singularities.

2. A complex hyperbolic cone manifold with codimension 2 orbifold singularities is in fact an orbifold.

3. Every finite complex hyperbolic orbifold is a lattice quotient.

I'll try to explain Theorem 1.2 in a different (but related) way which avoids the discussion of cone manifolds and most of the discussion of orbifolds. The objects I'll work with sound more technical, but in fact they are easier to understand because they refer very little to the structure of the singular locus. My route to Theorem 1.2 works like this:

1. Ignoring Conditions 2 and 3 above, the completion of $\mathcal{M}$ is always a finite volume complex hyperbolic stratified manifold with a fibered cone structure. Conditions 2 and 3 together imply that the completion of $\mathcal{M}$ has codimension 2 orbifold singularities.

2. Theorem 5.1 below: A complex hyperbolic stratified manifold with a fibered cone structure and codimension 2 orbifold singularities is a lattice quotient.

In §3, I'll explain the terms used above. In §4, I'll explain why one gets such aobjects from the details in Thurston's paper. In §5, I'll prove Theorem 5.1, thereby completing the proof of Theorem 1.2.

In §6 and 7, I'll prove Theorems 1.3 and 1.4 respectively. These results are essentially interpretations of Theorem 1.2. We just have to look back over the various constructions and see that they give the statements in Theorems 1.3 and 1.4.

2 Proof of Theorem 1.1

Let $\mathcal{M}$ be the labeled moduli space.

2.1 Step 1: Spanning Trees and Triangulations

A *embedded spanning tree* on a flat cone sphere Σ is spanning tree which has the cone points as vertices and no crossing edges. For instance, on the regular tetrahedron, the three edges emanating from a single vertex would be an embedded spanning tree.

Lemma 2.1 *Σ has an embedded spanning tree.*

Proof: First of all, Σ does have a spanning tree. One can connect every cone point to every other one by some straight line segment, and now one can choose a subgraph which is a spanning tree. Let τ be the spanning tree of minimum length. If a pair of edges in τ cross, then we can find a finite cycle $e_1, ..., e_k$ such that e_1 and e_k cross. But then we can switch the crossing, as shown in Figure 1. The switch gives a shorter spanning tree. ♠

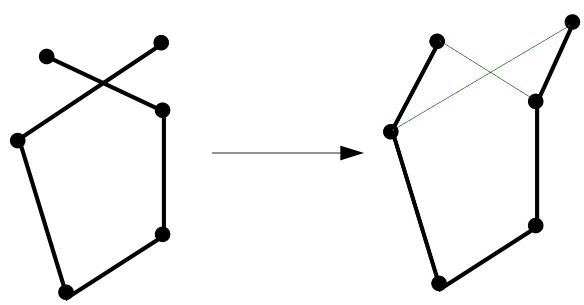

Figure 1: shortening the spanning tree.

For the proof of the next result, and for later purposes, say that a *pseudo-polygon* is a flat metric on a disk whose boundary is locally isometric to the boundary of a polygon. One typically gets a pseudo-polygon by immersing a polygon in the plane and pulling back the metric.

Lemma 2.2 *A pseudo-polygon P has a triangulation whose edges are straight line segments and whose vertices are the vertices of P.*

Proof: The proof goes by induction on the number of edges of P. The result is obvious if P has 3 edges. If P has more than 3 edges, then let v be a vertex of P and let e be an incident edge. Let L_t be a family of rays emanating from v so that L_0 extends v and the initial portion of L_t lies in P for $t > 0$ small. Let $S_t \subset L_t$ be the longest initial portion of L_t contained in the interior of P. Note that S_t is positive for all $t \in (0, \theta)$ where θ is the interior angle at v. There must be some $s \in (0, \theta)$ such that the endpoint of S_s is another vertex. But then S_s is an embedded segment connecting two distinct vertices of P. This segment divides P into two pseudo-polygons which both have triangulations by induction. ♠

Corollary 2.3 *Σ has a triangulation which just has the cone points as vertices.*

Proof: Let τ be an embedded spanning tree in Σ. The complement $\Sigma - \tau$ has a flat metric. The completion of this metric is a pseudo-polygon having twice as many edges as τ. But then we can triangulate this pseudo-polygon. ♠

Remark: Thurston proves Corollary 2.3 with a canonical construction using Voronoi cells and the dual Delaunay triangulation. However, I found it hard to see why this construction gives a triangulation rather than a union of triangles, with pairwise disjoint interiors, which perhaps only covers part of the surface. This is why I prefer the spanning tree approach. In class Saul Schliemer suggested that one could remove the vertices, pass to the universal cover, and then take the Delaunay triangulation there. That seems to work more convincingly than the argument in the paper, though I still prefer the spanning tree approach.

2.2 Step 2: Local Coordinates

Let Σ be a flat cone sphere, a point in $\mathcal{M}$. We are really interested in flat cone spheres mod similarity, but first we consider the larger space of flat cone structures. Let τ be an embedded spanning tree on Σ. A small neighborhood in $\mathcal{M}$ consists of flat cone spheres having an embedded spanning tree combinatorially identical to, and nearby, τ.

We orient the edges of τ in some way. Let P be the pseudo-polygon which is the completion of $\Sigma - \tau$. The developing map $D : P \to \boldsymbol{C}$ is well-defined because P is simply connected and has a flat metric. We label each edge e of P by the complex number

$$f(e) = D(e_+) - D(e_-). \tag{1}$$

Here e_+ is the head vertex of e and e_- is the tail vertex. Call this label $f(e)$. If e and e' are the two edges glued together, then we have a relation of the form $f(e') = u_e f(e)$, where u_e is some unit complex number that only depends on the list of cone deficits. One computes u_e by taking a loop in Σ which starts and ends at (say) the midpoint of e and avoids τ. This loop encloses some number of cone points, and the number u_e is $\exp(i\theta_e)$ where θ_e is either the sum of the cone deficits enclosed by the loop or 2π minus that sum. Which option depends on the orientation of the loop.

These labels make sense on all flat cone structures near Σ. If we multiply all labels by some complex number λ we get the same structure up to similarity. Moreover, the labels specify a pseudo-polygon which we can then glue together to get a point in $\mathcal{M}$. Thus, two nearby flat cone spheres are similar to each other if and only if their labels differ by this kind of scaling. In short, we have given local coordinate charts into projective space $\boldsymbol{CP}^{n-3}$. Here n is the number of cone points.

Our labels extend to give a system of labels on a triangulation of Σ extending τ. The labels on the remaining (oriented) edges are linear combinations of the labels of the edges of τ.

Suppose that we choose a different spanning tree. Each edge in the new tree cuts through a finite number of triangles of the old triangulation. When we develop these triangles out into the plane, we express the new edge label as some complex linear combination of old labels. Hence the changes of coordinates are complex linear. Remembering that we need to mod out by scaling, we see that the overlap functions for our charts are complex projective transformations. In particular, $\mathcal{M}$ is a complex projective manifold.

2.3 Step 3: The Hermitian Form

We fix a spanning tree τ on Σ and consider the local coordinates on the edges of the pseudo-polygon P. Call this larger space $\mathcal{P}$. A neighborhood of $\mathcal{M}$ about Σ is the quotient of $\mathcal{P}$ by scaling, as discussed above.

We have the area function $A : \mathcal{P} \to \boldsymbol{R}$. The area of a triangle spanned by edges z and w is

$$\pm \frac{i}{4}(z\overline{w} - w\overline{z}) \tag{2}$$

The sign depends on whether the vectors $\{z, w\}$ make a positively oriented or a negatively oriented basis.

When we express A as a function of the edge labels, we get a finite number of sums of terms like the one in Equation 2, where z and w are various complex linear combinations of the edge labels. Hence, A is the diagonal part of a Hermitian form. The coordinate changes are isometries relative to this form because changing the spanning tree does nothing to the area.

Now I'll explain why the Hermitian form has type $(1, n-2)$. But then the space $\mathcal{M}$ locally has the structure of $\boldsymbol{CP}^{n-3}$.

To explain the type of the Hermitian form, suppose that there are 2 cone deficits, say θ_1 and θ_2 such that $\theta_1 + \theta_2 < 2\pi$. Then we join the corresponding cone points by a straight line segment and slit Σ open along this line segment. We then glue in an appropriate portion of a cylinder to produce a new flat cone sphere Σ_{12} with one fewer cone point. Figure 2 shows a schematic view of this.

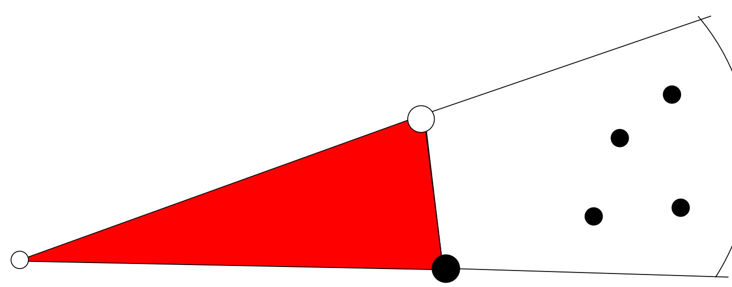

Figure 2: adding in the cone

Imagine that we have given linear coordinates $w_1, ..., w_{n-3}$ on Σ'. These coordinates tell us how to develop Σ' out into the plane. Let's say that the apex of the added (red) cone goes to the origin. Then there is some complex number z which describes the position of one image of θ_1 under the developing map. Fgure 3 shows what we are talking about.

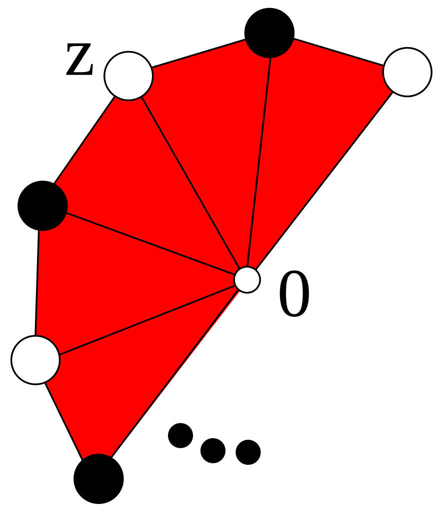

Figure 3: developing out the cone

Of course, the point z depends on which image under the developing map we choose. In general, we have countably many choices. However, once we make one choice for Σ, we can make the same choice, so to speak, for nearby structures. The coordinate z is a complex linear function of the linear coordinates on the space $\mathcal{X}$ described above. Thus, our coordinates $w_1, ..., w_{n-2}, z$ give linear coordinates on $\mathcal{X}$. In these coordinates, the function A has the form

$$A(w_1, ..., w_{n-2}, z) = A'(w_1, ..., w_{n-2}) - cz\overline{z}. \tag{3}$$

Here c is some constant which depends on the cone deficit of the added cone, and A' is the area form on the moduli space determined by the list $\theta_1 + \theta_2, \theta_3, ..., \theta_n$. Hence, if A' has type $(1, n-3)$ then A has type $(1, n-2)$.

We have done the induction step but not the base case. We can do a reduction above unless $n = 3$ or $n = 4$ and all the cone points have the same deficit. The case $n = 3$ is trivial – the moduli space is a single point. When $n = 4$ and all cone deficits are equal, the moduli space is a finite cover of the modular surface, which is modeled on $\boldsymbol{CH}^1$. So, the base cases work out.

3 Some Definitions

Stratified Manifolds: Let $\sqcup$ denote disjoint union. A *complex hyperbolic stratified manifold* is a complete metric space $X = X_0 \sqcup X_1 \sqcup X_2 \sqcup ...$ such

that

- X_0 is connected and has finite volume.
- X_k is locally isometric to $\boldsymbol{CH}^{n-k}$ for $k = 0, 1, 2, ...$
- $X_{k+1} \subset \text{closure}(X_k)$ for $k = 0, 1, 2, ...$

Fibered Cone Structure: We say that X is a *fibered cone manifold* if, for each k and each $r \in X_k$, we have some neighborhood N_r of r in X_0 with the following structure. First

$$N_r = \bigcup_{s \in \Delta} F_s. \tag{4}$$

Here Δ is an open metric disk about r in X_k. This is supposed to be a smooth fibration: There is a fiber preserving diffeomorphism $h : \Delta \times F_r \to N_r$. We call Δ a *basic disk*.

Second, ∂F_r is a spherical manifold, and F_r is foliated by geodesic arcs connecting points on ∂F_r to r. These arcs are all perpendicular to ∂F_r at their endpoints and they all have the same length – exactly the complex hyperbolic radius of the sphere on which ∂F_r is modeled.

Codimension 2 Conditions: $\boldsymbol{CH}^n$ contains $\boldsymbol{CH}^{n-1}$ as a totally geodesic submanifold. Let Y denote the universal cover of $\boldsymbol{CH}^n - \boldsymbol{CH}^{n-1}$. Let $\overline{Y}$ denote the metric completion of Y. The deck group extends to an action on $\overline{Y}$. The projection $\pi : \overline{Y} \to \boldsymbol{CH}^n$ is an infinite cyclic branched cover, branched over $\boldsymbol{CH}^{n-1}$.

For any $t > 0$ there is an isometry $I_t : \overline{Y} \to \overline{Y}$ which rotates $\overline{Y}$ by an angle of $2\pi t$ around $\overline{Y} - Y$. A *simple cone manifold* is a quotient of the form $\overline{Y}/I_t$ for $t \in (0, 1)$. When $1/t \in \boldsymbol{Z}$, we call Y/I_t a *simple orbifold*.

We say that a complex hyperbolic stratified manifold has codimension 2 simple cone (respectively orbifold) singularities if every point $p \in X_1$ has a neighborhood which is isometric to a ball in a simple cone manifold (respectively orbifold). The isometry needs to take X_1 into the singular set, and the dimensions are supposed to match up.

4 Structure of the Completion

Now I'll revisit Thurston's paper and explain why the completion of $\mathcal{M}$ has all the advertised properties. For ease of exposition, I'll work with the la-

beled space $\mathcal{M}$, under the assumption that the second angle condition simply does not occur. If the second angle condition does occur then we first do the analysis in the labeled case and then observe that the codimension 2 orbifold conditions emerge when we pass to the unlabeled quotient space. The point is that the relevant cone angle gets cut in half.

Stratified Structure: Say that a *multi-list* is a subset $\{\theta_{ij}\}$ of our list of deficits, with $i \in \{1, ..., k\}$, such that

$$\alpha_i = \sum_j \theta_{ij} < 2\pi, \qquad i = 1, ..., k \tag{5}$$

We can take a sequence of points in $\mathcal{M}$ corresponding to flat cone structures in which all the points corresponding to $\{\theta_{ij}\}$ for fixed i coalesce. The limit of this sequence is contained in a point in the completion of $\mathcal{M}$ corresponding to a stratum of codimension ℓ. Here $\ell + 3$ counts the number of cone points of the limiting flat cone spheres. This particular stratum is isometric to some lower dimensional moduli space $\mathcal{N}$. Moreover, any degeneration of structures in $\mathcal{M}$ arises this way. This gives the stratified structure.

Fibered Cone Structure: Associated to $\mathcal{N}$ is the multi-list above. Say that a cone point is *involved* if it is one of the points corresponding to our multi-list, and otherwise *uninvolved*. Suppose we hold the uninvolved points fixed and then coalesce the involved points. This produces a point $r \in \mathcal{N}$. There are k Euclidean cones $C_1, ..., C_k$ so that the jth cluster of involved points coalseces down to the apex of C_j. The point r corresponds to the flat cone structure defined by the apices of the C_j and the uninvolved points. Fixing the uninvolved points and varying the involved points gives F_r.

Why is F_r totally geodesic? As we did in the previous section, we can choose local linear coordinates so that the variables $w_1, w_2, ...$ describe the positions of the uninvolved points and the positions of the apices of the auxilliary cones, and then variables $z_1, z_2, ...$ describe the positions of the involved points. The points in $\boldsymbol{C}^{m-2}$ corresponding to F_r comprise a complex linear subspace. So, when we projectivize, we get some intersection of $\boldsymbol{CH}^{m-3}$ with a lower dimensional complex projective space. This gives us a totally geodesic copy of a lower dimensional complex hyperbolic place.

There is a natural foliation of F_r into arcs of geodesics. If we start with one flat cone structure corresponding to a point in F_r, we can move the jth cluster of involved points closer to the apex of C_j by a homothety (i.e.

similarity with no twisting). In terms of the coordinates we just mentioned, we are simply replacing $z_1, z_2, ...$ with $rz_1, rz_2, ...$ for some real $r < 1$. We fix some $A - \epsilon$, where A is the area of the flat cone surface associated to r (before rescaling) and ϵ is some small number. If we restrict F_r to points corresponding to structures having area in $(A - \epsilon, A)$ then ∂F_r is a spherical manifold.

There is a natural diffeomorphism from F_r to a nearby fiber F_s which comes from keeping the involved points fixed relative to the apices of the cones $C_1, ..., C_k$ and perturbing these apices and the uninvolved points. This map varies smoothly with s and gives rise to the smooth fibration structure.

Codimension 2 Conditions: Now we consider the codimension 2 strata. The simplest case occurs when $k = 1$ and $\{\theta_{ij}\}$ consists of just 2 deficits whose sum is less than 2π. Each choice leads to a connected (real) codimension 2 stratum. A local analysis, as done in class, shows that a neighborhood of each point on one of these strata is isometric to the simple cone manifolds Y/I_t discussed above. The value of t is $2\pi - \theta_{11} - \theta_{12}$. So, when Condition 1 on the deficits is satisfied, the corresponding stratum is a codimension 2 orbifold singularity.

A somewhat more subtle case occurs when $\theta_{11} = \theta_{12} = \theta$. The analysis applied to the labeled moduli space gives the angle around the stratum as $2\pi - 2\theta$. However, as mentioned above, when we pass to the unlabeled moduli space, we are taking a finite quotient which, in particular, cuts this cone angle in half. This case corresponds to Condition 2 on the deficits.

Finite Volume: (This part seems to be done just fine in Thurston's paper, and originally I hadn't said anything about it in these notes. But here I am adding some explanation.) Why does the space have finite volume? The non-compact ends of the the space correspond to paritions of the cone angles into two halves, each of which sum to π. There are finitely many partitions, and you want to see that each partition leads to a non-compact end with finite volume.

Fix one of these partitions. Outside a compact set, the corresponding cone manifolds is a "cigar" – a cylinder which has been capped off on either end. If you fix, say, the minimum distance between the cone points on the one end of the cigar and the cone points on the other, then the set of all cone structures realizing this minimum distance is compact and hence has finite volume.

These “fixed-minimum-distance” sets give a fibration of the non-compact end. There is a natural operation of inserting a cylinder in the middle (and rescaling the area). This insertion moves you from one fiber to another one further out. A local calculation shows that the insertion of a cylinder of length r decreases the volume of the fiber by $C\exp(-r)$, for some constant C. Hence, when you integrate over the fibers you get finite volume.

This argument is similar to the usual proof that a cusped complex hyperbolic manifold has finite volume.

5 Lattice Quotients

To finish the proof of Theorem 1.2, I'll prove the following result.

Theorem 5.1 *Let X be a complex hyperbolic stratified manifold with a fibered cone structure and codimension 2 orbifold conditions. Then X is a lattice quotient.*

The rest of these notes are devoted to proving Theorem 5.1. We make some basic definitions.

- Let $\widetilde{X}_0$ denote the universal cover of X_0.
- We have the developing map $D : \widetilde{X}_0 \to \boldsymbol{CH}^n$.
- We have the holonomy homomorphism $h : G = \pi_1(X_0) \to \mathrm{Isom}(\boldsymbol{CH}^n)$.
- Let $N \subset G$ be the kernel of the holonomy homomorphism.
- Let $\widehat{X}_0 = \widetilde{X}_0/N$.
- $\widehat{G} = G/N$.
- The developing map factors through a map $\widehat{D} : \widehat{X}_0 \to \boldsymbol{CH}^n$.

Let $\widehat{X}$ denote the metric completion of $\widehat{X}_0$.

Lemma 5.2 *The developing map $\widehat{D}$ extends to $\widehat{X}$ and is a distance non-increasing map. Also, $\pi : \widehat{X}_0 \to X_0$ extends to a map $\pi : \widehat{X} \to X$.*

Proof: Choose a point $p \in \widehat{X}$. There is a sequence $\{p_n\} \in \widehat{X}_0$ converging to p. Define $\widehat{D}(p) = \lim \widehat{D}(p_n) \in \boldsymbol{CH}^n$. This makes sense because $\widehat{D}$ is a local isometry on $\widehat{X}_0$ and hence distance non-increasing. In particular $\{\widehat{D}(p_n)\}$ is a Cauchy sequence. If $\{q_n\}$ converges to p as well then $d(p_n, q_n) \to 0$. But then we must have $d(\widehat{D}(p_n), \widehat{D}(q_n)) \to 0$ as well. Hence, $\widehat{D}(p)$ is well defined. Since $\widehat{D}$ is distance non-increasing on a dense subset of $\widehat{X}$, it is also distance non-increasing on $\widehat{X}$. The proof for π is essentially the same. ♠

Define

$$\widehat{X}_k = \pi^{-1}(X_k), \qquad k = 1, 2, 3, ... \tag{6}$$

The next result is where we use the codimension 2 orbifold conditions.

Lemma 5.3 (Removable Singularities) *Every point $p \in \widehat{X}_1$ has a neighborhood which is locally isometric to a ball in $\boldsymbol{CH}^n$ and the map $\widehat{D}$ gives such a local isometry.*

Proof: We first consider the picture in the space $\widetilde{X}$. All the constructions above made for $\widehat{X}$ also work for $\widetilde{X}$. Let C be a component of X_1 and let $\widetilde{C}$ be a corresponding component of $\widetilde{X}_1$. The codimension 2 cone manifold conditions tell us that the map $\pi : \widetilde{X} \to X$ is an infinite branched cover in a neighborhood of $\widetilde{C}$, branched over $\widetilde{C}$.

We know that neighborhoods of points in C are locally isometric to balls in Y/I_t for some $t = 1/k$. Let β be a small loop in $\widetilde{X}_0$ which winds k times around $\widehat{C}$. The element (really conjugacy class of elements) in the fundamental group $\pi_1(X_0)$ corresponding to β has trivial holonomy, and elements corresponding to loops winding fewer times around have nontrivial holonomy. For this reason, $\widetilde{X}$ is isometric to a neighborhood of $\overline{Y}/I_1 = \boldsymbol{CH}^n$ around $\widehat{C}$. ♠

The next lemma is where we use the fibered cone manifold conditions. This lemma seems obvious at first glance, because $\dim(X_k) = \dim(X) - 2k$. However, this seems like a slippery business. So, I am going to spell out the proof in a lot of detail.

Lemma 5.4 (Dimension) *$\widehat{D}(\widehat{X}_k)$ has codimension at least 4 for $k \geq 2$.*

Proof: Since X_k has a compact exhaustion, X_k is covered by countably many basic disks. Hence, it suffices to prove our result for $\widehat{\Delta} = \pi^{-1}(\Delta)$, where Δ is a basic disk. Let r be the center of Δ and let $N = N_r$ be the associated fibered neighborhood.

We claim that every point of $\widehat{p} \in \widehat{\Delta}$ is an accumulation point of a path component of $\pi^{-1}(N \cap X_0)$. To see this, let $\{\widehat{q}_n\}$ be a Cauchy sequence in $\widehat{X}_0$ converging to $\widehat{p}$. The metric on $\widehat{X}_0$ is the path metric, so we can find a path $\widehat{\gamma}_{mn}$ joining $\widehat{q}_m$ to $\widehat{q}_n$ which is within a factor of 2 of the actual distance in $\widehat{X}_0$ between $\widehat{q}_m$ and $\widehat{q}_n$. Since π does not increase distances, $\{q_n\}$ is a Cauchy sequence in X_0 converging to p. But then $q_n \in N$ for large n. Moreover $\gamma_{mn} \subset N$ for large m, n. But then the entire path $\widehat{\gamma}_{mn}$ stays in the same path component of $\pi^{-1}(N \cap X_0)$. Hence, the tail end of our Cauchy sequence $\{\widehat{q}_n\}$ stays in the same path component. This establishes the claim.

The space $\widehat{X}_0$ contains a countable dense set, and each path component of $\pi^{-1}(N \cap X_0)$ is an open set containing one point in this dense set that is not contained in any of the others. Therefore, there are only countable many components of $\pi^{-1}(N \cap X_0)$. In light of our claim above, it suffices to prove, for an arbitrary path component $\widehat{A}$ of $\pi^{-1}(N \cap X_0)$, that $\widehat{D}(\widehat{\Delta} \cap \text{closure}(\widehat{A}))$ has dimension $2n - 2k$ in $\boldsymbol{CH}^n$.

Since N is foliated by sets of the form F_s for $s \in \Delta$, we have the decomposition

$$\widehat{A} = \bigcup_{s \in \Delta} \widehat{A}_s, \qquad \widehat{A}_s = \pi^{-1}(F_s \cap X_0). \tag{7}$$

Since π is a local isometry on $\widehat{X}_0$, we see that $\widehat{A}_s$ is foliated by geodesic arcs, all of the same length, which meet $\partial \widehat{A}_s$ at right angles. Here $\widehat{A}_s$ is a manifold modeled on a complex hyperbolic sphere. The foliating arcs give a retraction of $\widehat{A}$ onto its ∂A. (Remember the the inner endpoints of the foliating arcs are not part of $\widehat{A}$.) Hence $\partial \widehat{A}$ is connected. At the same time, the product structure on $N \cap X_0$ gives a continuous retraction from $N \cap X_0$ to each fiber $F_s \cap X_0$. This continuous rectraction lifts to a continuous rectraction from $\partial \widehat{A}$ to $\partial \widehat{A}_s$. Hence $\partial \widehat{A}_s$ is connected. Therefore, the image $\widehat{D}(\partial \widehat{A}_s)$ is contained in a geodesic sphere S_s in $\boldsymbol{CH}^n$. Moreover, $\widehat{D}$ maps the geodesic arcs foliating $\widehat{A}_s$ to geodesic arcs perpendicular to S_s and pointing inward. These geodesic arcs all have the same length, so they all meet at the center c_s of S_s.

Suppose that we have a Cauchy sequence $\{\widehat{q}_n\}$ converging to some point $\widehat{s} \in \widehat{\Delta} \cap \text{closure}(\widehat{A})$. Then $\widehat{q}_n$ lies on some foliating arc of some $\widehat{A}_{s_n}$. Since

there is a minimum positive distance between s and any fiber F_t with $t \neq s$, we must have $s_n \to s$. Moreover, when n is large, $\widehat{q}_n$ lies almost all the way at the inner end of the foliating arc. Hence, the distance from $\widehat{D}(\widehat{q}_n)$ to c_s tends to 0 as n tends to ∞. Hence

$$\widehat{D}(\widehat{\Delta} \cap \text{closure}(\widehat{A})) \subset Y = \bigcup_{s \in \Delta} c_s \tag{8}$$

Given the smooth nature of the fibration, the point c_s varies smoothly with $s \in \Delta$. This shows that Y is a smooth manifold of dimension $2n - 2k$. ♠

Lemma 5.5 $\widehat{D}(\widehat{X}) = \boldsymbol{CH}^n$.

Proof: From Lemma 5.3, the map $\widehat{D}$ is a local isometry from $\widehat{X}_0 \cup \widehat{X}_1$ to $\boldsymbol{CH}^n$. Suppose $\widehat{D} : \widehat{X} \to \boldsymbol{CH}^n$ is not onto. Let $q \in \boldsymbol{CH}^n - \widehat{D}(\widehat{X})$. Pick $p \in \widehat{D}(\widehat{X}_0 \cup \widehat{X}_1)$ and consider the geodesic γ connecting p to q. Choosing p generically and using the Dimension Lemma, we can arrange that γ does not intersect $\widehat{D}(\widehat{X}_k)$ for $k \geq 2$.

Let $\widehat{p}$ be some pre-image of p in $\widehat{X}_0 \cup \widehat{X}_1$. There is some initial geodesic segment $\widehat{\alpha}$ which $\widehat{D}$ carries to the initial portion of γ emanating from p. The geodesic $\widehat{\gamma}$ extending $\widehat{\alpha}$ lies entirely in $\widehat{X}_0 \cup \widehat{X}_1$, by construction. But then $\widehat{D}$ is defined on all of $\widehat{\gamma}$ and in particular $q \in \widehat{D}(\widehat{\gamma}) \subset \widehat{D}(\widehat{X}_0 \cup \widehat{X}_1)$. This is a contradiction. ♠

Lemma 5.6 *$\widehat{D}$ is injective on $\widehat{X}_0 \cup \widehat{X}_1$.*

Proof: By Lemma 5.5, the map $\widehat{D} : \widehat{X}_0 \cup \widehat{X}_1 \to \boldsymbol{CH}^n$ is a local isometry and therefore a covering map of its image. But, by the Dimension Lemma and Lemma 5.5, the image $\widehat{D}(\widehat{X}_0 \cup \widehat{X}_1)$ is everything but a set of codimension at least 4. Hence $\widehat{D}(\widehat{X}_0 \cup \widehat{X}_1)$ is simply connected. But then our covering map must be injective. ♠

$\widehat{D}$ is a global isometry from $\widehat{X}_0 \cup \widehat{X}_1$ to an open dense subset of $\boldsymbol{CH}^n$. So, we can identify $\widehat{X}_0 \cup \widehat{X}_1$ with an open dense subset of $\boldsymbol{CH}^n$. We make this identification. Let $\Gamma = \widehat{G}$. Under our identification, $\boldsymbol{CH}^n$ is the metric completion of $\widehat{X}_0 \cup \widehat{X}_1$. Hence $\widehat{X} = \boldsymbol{CH}^n$. But then Γ acts isometrically

on $\boldsymbol{CH}^n$. The action is discrete and co-finite because $\widehat{X}_0/\Gamma = X_0$ has finite volume and nonempty interior. Hence Γ is a lattice. Since X_0 is dense in $\boldsymbol{CH}^n/\Gamma$, and $\boldsymbol{CH}^n/\Gamma$ has finite volume, we see that $\boldsymbol{CH}^n/\Gamma$ is the metric completion of X_0. Hence $X = \boldsymbol{CH}^n/\Gamma$. This completes the proof.

6 Proof of Theorem 1.3

I'll prove Theorem 1.3 through a series of smaller results.

Lemma 6.1 *If $\mathcal{M}$ is special, the set of triangulation points is dense in $\mathcal{M}$.*

Proof: Let's look at the local coordinates we get when we have a triangulation. We take some embedded spanning tree and associate the coordinates as above. When we cut along the spanning tree and look at the resulting pseudo-polygon, we can develop it into $\boldsymbol{C}$ so that the vertices lie in **Eis**. Moreover, the unit complex numbers $\{u_i\}$ relating pairs of coordinates (on edges which get glued together) also belong to **Eis**. In short, all the coordinates lie in **Eis**. Conversely, if we choose sufficiently nearby coordinates in **Eis**, we get a triangulation.

Now, if we have any flat cone sphere corresponding to a point in **M**, we can scale it up so that it has enormous coordinates with respect to some spanning tree, and then we can find nearby coordinates in **Eis** which just differ by at most 2 units from the original coordinates. When we scale back down to (say) unit area, the original structure and the nearby triangulation point are extremely close. Hence the triangulation points are dense in **M**. ♠

Lemma 6.2 *If $\mathcal{M}$ is special, then the completion of $\mathcal{M}$ is a lattice quotient.*

Proof: We just have to verify the deficit conditions. We have $\theta_i = k_i\pi/3$ for some $k_i \in \{1,2,3\}$. If $\theta_i + \theta_j < 2\pi$ and $\theta_i \neq \theta_j$ then $2\pi - \theta_i - \theta_j = k_{ij}\pi/3$ for some $k_{ij} = 1,2,3$. Hence, the first condition on the deficits holds. If $\theta_i = \theta_j$ and $\theta_i < \pi$ then, again 2π is an integer multiple of $\pi - \theta_i$. Now we apply Theorem 1.2. ♠

Let Γ be the lattice such that the completion of $\mathcal{M}$ is $\boldsymbol{CH}^{m-3}/\Gamma$. Now, Γ acts as a group of matrices on $\boldsymbol{C}^{m-2}$ and preserves some Hermitian form

A of type $(1, m-2)$. To get actual matrices, we need to choose some linear coordinates on $\boldsymbol{C}^{m-2}$. We choose the coordinates coming from the embedded spanning trees. This gives $2m-2$ variables, but we choose $m-2$ independent ones.

Lemma 6.3 *With respect to the coordinates coming from the embedded spanning trees, the entries of elements of* Γ *all lie in* **Eis**.

Proof: Suppose we start with a closed loop in $\mathcal{M}$. We develop $\mathcal{M}$ into $\boldsymbol{CH}^{m-2}$ along this loop and then take the holonomy. This gives us some element of Γ, and all elements of Γ arise this way.

We can break our loop into finitely many segments, such that each segment is contained in a single spanning tree coordinate chart on $\mathcal{M}$. As we move from segment to segment, we make some linear change of coordinates. The element of Γ is the product of these coordinate-change matrices.

Now, when we compute the coordinate change matrices, we can compute them with respect to triangulation points, because the triangulation points are dense. But, from the description of the coordinate changes given in §2.2 we see that each coordinate on the new spanning tree is a complex linear combination of the old coordinates, where the coefficients of the linear combination lie in **Eis**. Hence, the coordinate change matrices have entries in **Eis**. Hence, so does the product of these matrices. ♠

Lemma 6.4 Γ *preserves a Hermitian form* H *of type* $(1, 9)$ *which is defined over* **Eis**.

Proof: Choose some point $p \in \mathcal{M}$ and let Σ be the corresponding flat cone sphere. Let τ be some embedded spanning tree on Σ. When we develop out $\Sigma - \tau$ into the plane, we get $2n-2$ coordinates which, in pairs, are related by complex numbers $u_1, ..., u_{n-1}$. But, due to the values of the cone angles, these numbers are all 6th roots of unity. They all belong to **Eis**. So, when we triangulate the pseudo-polygon $\Sigma - \tau$, the other labels are complex linear combinations of the original variables, with coefficients in **Eis**. ♠

Proof of Theorem 1.3: The lattice Γ corresponds to the moduli space which contains the regular icosahedral tiling. This lattice acts on $\boldsymbol{CH}^9$ because there are 12 deficits on the list and $9 = 12 - 3$. If we coalesce various of

the cone points corresponding to the regular icosahedron, we can achieve a deficit list corresponding to every other type of triangulation. (For instance, if we coalesce the points in pairs we get triangulations having the same deficit list as the list produced by the octahedron.) But this means that every special moduli space is some stratum of $\boldsymbol{CH}^9/\Gamma$. From what we have already seen, Γ is defined over **Eis**. ♠

7 Proof of Theorem 1.4

Let Γ be the lattice from Theorem 1.3. We fix some point in $\mathcal{M}$, say the structure corresponding to the regular icosahedron. We also fix some embedded spanning tree τ relative this structure. There is some open cone $\mathcal{C}$ in $\boldsymbol{C}^{1,9}$ such that points in $\mathcal{C}$ correspond, via coordinates on τ, to some open set in $\mathcal{M}$. We define the Hermitian form $H = 4\sqrt{3}A$ with respect to τ. The elements of Γ preserve both H and $\mathbf{Eis}^{1,9}$, even though they typically move the cone $\mathcal{C}$ off itself.

Lemma 7.1 *H is defined by a metrix with entries in* **Eis**. *Given a positive vector $V \in \mathcal{C} \cap \mathbf{Eis}^{1,9}$, the norm $H(V,V)$ computes the 4 times the area of the flat cone sphere.*

Proof: When we work out the formula for H with respect to τ, we see that it just involves expressions of the form $\sqrt{3}i(z\overline{w} - w\overline{z})$, where z and w are complex linear combinations of the coordinates with coefficients in **Eis**. This easily implies that H is defined over **Eis**.

For the second statement, we observe that a unit equilateral triangle has area $\sqrt{3}/4$. Hence the quantity $H(V,V)$, which records $4\sqrt{3}$ times the area, counts 3 times the number of triangles. ♠

From Triangulations to Vectors: For any pair (Σ, σ), where Σ is a flat cone sphere and σ is an embedded spanning tree, there is some finite sequence of coordinate changes whose composition allows us to express the σ-coordinates as τ-coordinates. What we have is a finite sequence (Σ_j, σ_j) and a finite sequence $\mathcal{C}_j$ of cones such that each point in $\mathcal{C}_j$ corresponds, via coordinates on σ_j to a flat cone sphere. Here $j = 0, ..., k$ and $\sigma_0 = \sigma$ and $\sigma_k = \tau$. The cones $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ overlap, and there is some matrix M_{j+1},

defined over **Eis**, which expresses the coordinate changes on the overlap. The product of the matrices $M = M_k ... M_1$ expresses the σ-coordinates in terms of the τ coordinates, even though the τ coordinates may not lie in the cone $\mathcal{C} = \mathcal{C}_k$. This does not bother us. The important point is that these coordinate changes preserve both $\mathbf{Eis}^{1,9}$ and H.

Now, suppose we were to take a different sequence (Σ'_j, σ'_j) for $j = 1, ..., \ell$, with $\sigma'_0 = \sigma$ and $\sigma'_\ell = \tau$. This would give us sequence of cones and matrices, and hence a new coordinate change. In this case, we would apply the matrix $M' = M'_\ell ... M'_1$ to the σ-coordinates. The matrix $M' \circ M^{-1}$ is the result of doing a "loop of coordinate changes" starting and ending at (Σ, τ). Hence, this matrix belongs to Γ. In short, any two of our coordinate changes differ by the action of Γ. In other words, given σ-coordinates, there is a canonical point in $\boldsymbol{C}^{9,1}/\Gamma$ that represents the τ-coordinates mod Γ.

So, if we start with a triangulation of the sphere, we get a point in $\mathbf{Eis}^{1,9}$ relative to some embedded spanning tree. We then make a coordinate change and get a well-defined positive vector in $\mathbf{Eis}^{1,9}/\Gamma$.

From Vectors to Triangulations: Conversely, if we have a positive vector in $\mathbf{Eis}^{1,9}/\Gamma$ we take some representative vector $V \in \mathbf{Eis}^{1,9}$ and then interpret V as giving coordinates relative to our preferred tree τ, even though these coordinates might not lie in the cone $\mathcal{C}$. We then make a finite string of coordinate changes until we arrive at a new vector $W \in \mathbf{Eis}^{1,9}$ giving coordinates relative to a spanning tree σ which is embedded on the flat cone sphere Σ corresponding to $[W]$. This gives us a triangulation of the sphere.

This triangulation is independent of our choice of coordinate change, and also independent of the choice of V. If we make the construction twice, the two triangulations on Σ have σ-coordinates which differ by a loop of coordinate changes, as above, starting from and ending at (Σ, σ). These coordinate changes do nothing to the triangulation.

The two halves of our construction are inverses of each other, so we get a bijection between the advertised sets. Since Γ is constructed out of the kind of sequences of coordinate changes discussed above, Γ preserves the Hermitian form H. As we have already mentioned, $H(V, V)$ counts 3 times the number of triangles.

8 References

Here are some additional refrences:

- Curt McMullen's paper *The Gauss-Bonnet Theorem for Cone Manifolds and Volumes of Moduli Spaces.* works out a general theory of cone manifolds which adds details to Thurston's description, especially a prime factorization theorem for cone manifolds.

- M. Weber's thesis 1993 Bonn thesis *Fundamentalbereiche komplex hyperbolischer Fl ächen.* works through Thurston's construction using star-shaped spanning trees. I don't know how well this matches what I do above.

- John Parker's paper *J.R. Parker, Cone metrics on the sphere and Livn's lattices* (Acta Mathematica 196 (2006) 1-64) Works through Thurston's construction explicitly for Livné's lattices, including building fundamental polyhedra using the Poincaré theorem. The Livné's lattices are special cases, corresponding to moduli spaces with 5 cone points.

- An upcoming paper by John Parker and Richard K Boadi, *Mostow's lattices and cone metrics on the sphere* (Advances in Geometry) does the same thing as Parker's earlier paper but for some of Mostow's lattices.